Propagation of regularity in L^p-spaces for Kolmogorov type hypoelliptic operators

Xicheng Zhang

Wuhan University

(A joint work with Zhen-Qing Chen)

Cheng-Du•2018.7.16

Introduction

- Let $n \geqslant 2$ and $d \in \mathbb{N}$.

Introduction

- Let $n \geqslant 2$ and $d \in \mathbb{N}$.
- Consider the following Kolmogorov type hypoelliptic operator:

$$\mathscr{L}_t := \sum_{i,j=1}^{d} a_t^{ij} \partial_{x_{ni}} \partial_{x_{nj}} + \sum_{j=2}^{n} x_j \cdot \nabla_{x_{j-1}}, \tag{1.1}$$

where $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^{nd}$ with $x_j = (x_{j1}, \cdots, x_{jd}) \in \mathbb{R}^d$, $\nabla_{x_j} = (\partial_{x_{j1}}, \cdots, \partial_{x_{jd}})$, $a_t = (a_t^{ij}) : \mathbb{R} \to \mathbb{M}^d_{sym}$ is a measurable map.

Introduction

- Let $n \geqslant 2$ and $d \in \mathbb{N}$.
- Consider the following Kolmogorov type hypoelliptic operator:

$$\mathscr{L}_t := \sum_{i,j=1}^{d} a_t^{ij} \partial_{x_{ni}} \partial_{x_{nj}} + \sum_{j=2}^{n} x_j \cdot \nabla_{x_{j-1}}, \tag{1.1}$$

where $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^{nd}$ with $x_j = (x_{j1}, \cdots, x_{jd}) \in \mathbb{R}^d$, $\nabla_{x_j} = (\partial_{x_{j1}}, \cdots, \partial_{x_{jd}})$, $a_t = (a_t^{ij}) : \mathbb{R} \to \mathbb{M}^d_{sym}$ is a measurable map.

- Here $\mathbb{M}^d_{sym}$ stands for the set of all symmetric $d \times d$-matrices. Suppose that for some $\kappa \geqslant 1$,

$$\kappa^{-1} \mathbb{I}_{d\times d} \leqslant a_t \leqslant \kappa \mathbb{I}_{d\times d}. \tag{1.2}$$

- Let $\nabla := (\nabla_{x_1}, \cdots, \nabla_{x_n})$, $\nabla^2_{x_n} := (\partial_{x_{ni}}\partial_{x_{nj}})_{i,j=1,\cdots,d}$ and

$$A = A_n = \begin{pmatrix} 0_{d\times d} & \mathbb{I}_{d\times d} & \cdots & \cdots & 0_{d\times d} \\ 0_{d\times d} & 0_{d\times d} & \mathbb{I}_{d\times d} & 0_{d\times d} & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & 0_{d\times d} & \mathbb{I}_{d\times d} \\ 0_{d\times d} & \cdots & \cdots & 0_{d\times d} & 0_{d\times d} \end{pmatrix}_{nd\times nd}. \tag{1.3}$$

- Let $\nabla := (\nabla_{x_1}, \cdots, \nabla_{x_n})$, $\nabla^2_{x_n} := (\partial_{x_{ni}}\partial_{x_{nj}})_{i,j=1,\cdots,d}$ and

$$A = A_n = \begin{pmatrix} 0_{d\times d} & \mathbb{I}_{d\times d} & \cdots & \cdots & 0_{d\times d} \\ 0_{d\times d} & 0_{d\times d} & \mathbb{I}_{d\times d} & 0_{d\times d} & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & 0_{d\times d} & \mathbb{I}_{d\times d} \\ 0_{d\times d} & \cdots & \cdots & 0_{d\times d} & 0_{d\times d} \end{pmatrix}_{nd\times nd}. \tag{1.3}$$

- We can rewrite $\mathscr{L}_t$ as the following compact form:

$$\mathscr{L}_t = \mathrm{tr}(a_t \cdot \nabla^2_{x_n}) + Ax \cdot \nabla,$$

where “tr” denotes the trace of matrix.

- Consider the following linear stochastic differential equations (SDEs):

$$\mathrm{d}X_t^{s,x} = AX_t^{s,x}\mathrm{d}t + \sigma_t^a \mathrm{d}W_t \quad \text{for } t > s \ \text{ with } X_s^{s,x} = x, \tag{1.4}$$

where $(W_t)_{t\in\mathbb{R}}$ is a standard *nd*-dimensional Brownian motion and

$$\sigma_t^a := \begin{pmatrix} 0_{(n-1)d\times(n-1)d}, & 0_{(n-1)d\times d} \\ 0_{d\times(n-1)d}, & (\sqrt{2a_t})_{d\times d} \end{pmatrix}_{nd\times nd}. \tag{1.5}$$

- Consider the following linear stochastic differential equations (SDEs):

$$\mathrm{d}X_t^{s,x} = AX_t^{s,x}\mathrm{d}t + \sigma_t^a \mathrm{d}W_t \quad \text{for } t > s \ \text{ with } X_s^{s,x} = x, \tag{1.4}$$

where $(W_t)_{t\in\mathbb{R}}$ is a standard nd-dimensional Brownian motion and

$$\sigma_t^a := \begin{pmatrix} 0_{(n-1)d\times(n-1)d}, & 0_{(n-1)d\times d} \\ 0_{d\times(n-1)d}, & (\sqrt{2a_t})_{d\times d} \end{pmatrix}_{nd\times nd}. \tag{1.5}$$

- If $a_t = a$ does not depend on t (i.e., time homogeneous), then

$$X_t^{s,x} \overset{(d)}{=} Z_{t-s}^x \ \text{ with } \ Z_t^x := \mathrm{e}^{tA}x + \int_0^t \mathrm{e}^{rA}\sigma^a \mathrm{d}W_r. \tag{1.6}$$

- Z_t^x is an (nd)-dimensional Gaussian random variable with density

$$p_t(x,y) = \frac{\mathrm{e}^{-(\Theta_{t^{-1/2}}(y-\mathrm{e}^{tA}x))^*\Sigma^{-1}\Theta_{t^{-1/2}}(y-\mathrm{e}^{tA}x)}}{((2\pi)^{nd}t^{n^2d}\det(\Sigma))^{1/2}},$$

where $\Theta_r : \mathbb{R}^{nd} \to \mathbb{R}^{nd}$ is the dilation operator defined by

$$\Theta_r(x) = (r^{2n-1}x_1, r^{2n-3}x_2, \cdots, rx_n), \tag{1.7}$$

and $\Sigma := \int_0^1 \mathrm{e}^{rA}\sigma^a(\sigma^a)^*\mathrm{e}^{rA^*}\mathrm{d}r$ is the covariance matrix of Z.

- Z^x_t is an (nd)-dimensional Gaussian random variable with density

$$p_t(x,y) = \frac{\mathrm{e}^{-(\Theta_{t^{-1/2}}(y-\mathrm{e}^{tA}x))^*\Sigma^{-1}\Theta_{t^{-1/2}}(y-\mathrm{e}^{tA}x)}}{((2\pi)^{nd}t^{n^2d}\det(\Sigma))^{1/2}},$$

where $\Theta_r : \mathbb{R}^{nd} \to \mathbb{R}^{nd}$ is the dilation operator defined by

$$\Theta_r(x) = (r^{2n-1}x_1, r^{2n-3}x_2, \cdots, rx_n), \tag{1.7}$$

and $\Sigma := \int_0^1 \mathrm{e}^{rA}\sigma^a(\sigma^a)^*\mathrm{e}^{rA^*}\mathrm{d}r$ is the covariance matrix of Z.

- For $f \in C^2_b(\mathbb{R}^{nd})$, define

$$\mathcal{T}_{s,t}f(x) := \mathbb{E}f(X^{s,x}_t). \tag{1.8}$$

- Z_t^x is an (nd)-dimensional Gaussian random variable with density

$$p_t(x,y) = \frac{\mathrm{e}^{-(\Theta_{t^{-1/2}}(y-\mathrm{e}^{tA}x))^*\Sigma^{-1}\Theta_{t^{-1/2}}(y-\mathrm{e}^{tA}x)}}{((2\pi)^{nd}t^{n^2d}\det(\Sigma))^{1/2}},$$

where $\Theta_r : \mathbb{R}^{nd} \to \mathbb{R}^{nd}$ is the dilation operator defined by

$$\Theta_r(x) = (r^{2n-1}x_1, r^{2n-3}x_2, \cdots, rx_n), \tag{1.7}$$

and $\Sigma := \int_0^1 \mathrm{e}^{rA}\sigma^a(\sigma^a)^*\mathrm{e}^{rA^*}\mathrm{d}r$ is the covariance matrix of Z.

- For $f \in C_b^2(\mathbb{R}^{nd})$, define

$$\mathcal{T}_{s,t}f(x) := \mathbb{E}f(X_t^{s,x}). \tag{1.8}$$

- $\mathscr{L}_t$ is the infinitesimal generator of $\mathcal{T}_{s,t}$.

$$\partial_s\mathcal{T}_{s,t}f + \mathscr{L}_s\mathcal{T}_{s,t}f = 0. \tag{1.9}$$

- Define

$$u(s,x) := \int_0^\infty \mathrm{e}^{-\lambda t}\mathcal{T}_{s,t+s}f(t+s,x)\mathrm{d}t = \int_s^\infty \mathrm{e}^{-\lambda(r-s)}\mathcal{T}_{s,r}f(r,x)\mathrm{d}r.$$

- Define

$$u(s,x) := \int_0^\infty \mathrm{e}^{-\lambda t}\mathcal{T}_{s,t+s}f(t+s,x)\mathrm{d}t = \int_s^\infty \mathrm{e}^{-\lambda(r-s)}\mathcal{T}_{s,r}f(r,x)\mathrm{d}r.$$

- One sees that

$$\partial_s u(s,x) + (\mathscr{L}_s - \lambda)u(s,x) + f(s,x) = 0. \tag{1.10}$$

- Define

$$u(s,x) := \int_0^\infty \mathrm{e}^{-\lambda t}\mathcal{T}_{s,t+s}f(t+s,x)\mathrm{d}t = \int_s^\infty \mathrm{e}^{-\lambda(r-s)}\mathcal{T}_{s,r}f(r,x)\mathrm{d}r.$$

- One sees that

$$\partial_s u(s,x) + (\mathscr{L}_s - \lambda)u(s,x) + f(s,x) = 0. \qquad (1.10)$$

- Our main result is

Theorem 1 (Local version)

Let $p \in (1,\infty)$. Under the uniform ellipticity condition (1.2), there is a constant $C = C(n,\kappa,p,d) > 0$ such that for all $f \in L^p$ and $\lambda \geqslant 0$,

$$\left\|\Delta_{x_j}^{1/(1+2(n-j))}u\right\|_p \leqslant C\|f\|_p, \quad j = 1,\dots,n, \qquad (1.11)$$

where $\Delta_{x_j}^{1/(1+2(n-j))} := -(-\Delta_{x_j})^{1/(1+2(n-j))}$ is the fractional Laplacian acting on the j-th variable $x_j \in \mathbb{R}^d$.

- Consider the following $n+1$-order stochastic differential equation:

$$\mathrm{d}X_t^{(n)} = b_t(X_t, X_t^{(1)}, \cdots, X_t^{(n)})\mathrm{d}t + \sigma_t(X_t, X_t^{(1)}, \cdots, X_t^{(n)})\mathrm{d}\widetilde{W}_t,$$

- Consider the following $n+1$-order stochastic differential equation:

$$\mathrm{d}X_t^{(n)} = b_t(X_t, X_t^{(1)}, \cdots, X_t^{(n)})\mathrm{d}t + \sigma_t(X_t, X_t^{(1)}, \cdots, X_t^{(n)})\mathrm{d}\widetilde{W}_t,$$

- $X_t^{(n)}$ denotes the n-order derivative of X_t in the time variable.

- Consider the following $n+1$-order stochastic differential equation:

$$\mathrm{d}X_t^{(n)} = b_t(X_t, X_t^{(1)}, \cdots, X_t^{(n)})\mathrm{d}t + \sigma_t(X_t, X_t^{(1)}, \cdots, X_t^{(n)})\mathrm{d}\widetilde{W}_t,$$

- $X_t^{(n)}$ denotes the n-order derivative of X_t in the time variable.
- $b : \mathbb{R}_+ \times \mathbb{R}^{(n+1)d} \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^{(n+1)d} \to \mathbb{R}^d \otimes \mathbb{R}^d$ are measurable functions.

- Consider the following $n+1$-order stochastic differential equation:

$$\mathrm{d}X_t^{(n)} = b_t(X_t, X_t^{(1)}, \cdots, X_t^{(n)})\mathrm{d}t + \sigma_t(X_t, X_t^{(1)}, \cdots, X_t^{(n)})\mathrm{d}\widetilde{W}_t,$$

- $X_t^{(n)}$ denotes the n-order derivative of X_t in the time variable.
- $b : \mathbb{R}_+ \times \mathbb{R}^{(n+1)d} \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^{(n+1)d} \to \mathbb{R}^d \otimes \mathbb{R}^d$ are measurable functions.
- $\widetilde{W}_t$ is a d-dimensional Brownian motion.

- Let

$$\mathbf{X}_t := (X_t, X_t^{(1)}, \cdots, X_t^{(n)}).$$

- Let

$$\mathbf{X}_t := (X_t, X_t^{(1)}, \cdots, X_t^{(n)}).$$

- $\mathbf{X}_t$ solves the following one order stochastic differential equation

$$\mathrm{d}\mathbf{X}_t = (X_t^{(1)}, \cdots, X_t^{(n)}, b_t(\mathbf{X}_t))\mathrm{d}t + (0, \cdots, 0, \sigma_t(\mathbf{X}_t)\mathrm{d}\widetilde{W}_t), \quad \mathbf{X}_0 = \mathbf{x},$$

where $\mathbf{x} = (\mathbf{x}_i)_{i=0,\cdots,n} = ((x_{ij})_{j=1,\cdots,d})_{i=0,\cdots,n}$.

- Let
$$\mathbf{X}_t := (X_t, X_t^{(1)}, \cdots, X_t^{(n)}).$$

- $\mathbf{X}_t$ solves the following one order stochastic differential equation
$$\mathrm{d}\mathbf{X}_t = (X_t^{(1)}, \cdots, X_t^{(n)}, b_t(\mathbf{X}_t))\mathrm{d}t + (0, \cdots, 0, \sigma_t(\mathbf{X}_t)\mathrm{d}\widetilde{W}_t), \quad \mathbf{X}_0 = \mathbf{x},$$
where $\mathbf{x} = (\mathbf{x}_i)_{i=0,\cdots,n} = ((x_{ij})_{j=1,\cdots,d})_{i=0,\cdots,n}$.

- In particular, the infinitesimal generator of Markov process $\mathbf{X}_t(\mathbf{x})$ is given by
$$\mathscr{L}_t f(\mathbf{x}) = \sum_{i,j,k=1}^{d} (\sigma_t^{ik}\sigma_t^{jk})(\mathbf{x})\partial_{x_{ni}}\partial_{x_{nj}} f(\mathbf{x}) + \sum_{j=1}^{n} \mathbf{x}_j \cdot \nabla_{\mathbf{x}_{j-1}} f(\mathbf{x}) + b_t(\mathbf{x}) \cdot \nabla_{\mathbf{x}_n} f(\mathbf{x}).$$

- Estimate (1.11) could be used to study the well-posedness of the above SDE with rough coefficients b and σ.

- Estimate (1.11) could be used to study the well-posedness of the above SDE with rough coefficients b and σ.
- When $n = 1$ and σ is bounded and uniformly nondegenerate, in [1] we have studied the strong well-posedness with both $(\mathbb{I} - \Delta_{\mathbf{x_1}})^{1/3} b$ and $\nabla\sigma$ in $L^p_{loc}(\mathbb{R}_+ \times \mathbb{R}^{2d})$ for some $p > 4d + 2$.

- Estimate (1.11) could be used to study the well-posedness of the above SDE with rough coefficients b and σ.
- When $n = 1$ and σ is bounded and uniformly nondegenerate, in [1] we have studied the strong well-posedness with both $(\mathbb{I} - \Delta_{\mathbf{x_1}})^{1/3} b$ and $\nabla\sigma$ in $L^p_{loc}(\mathbb{R}_+ \times \mathbb{R}^{2d})$ for some $p > 4d + 2$.
- Fedrizzi E., Flandoli, Priola and Vovelle[2] obtained the similar results when $\sigma_t = \mathbb{I}_{d\times d}$.

[1] Zhang X.: Stochastic Hamiltonian flows with singular coefficients. *Science China: Mathematics* (2018+).

[2] Fedrizzi, Flandoli F., Priola E. and Vovelle J.: Regularity of Stochastic Kinetic Equations. *Electron. J. Probab.* Volume 22 (2017), paper no. 48, 42 pp.

- Bramanti, Cupini, Lanconelli and Priola[1] adopted Coifman-Weiss' theorem to show the estimate (1.11) for $j = n$.

- Bramanti, Cupini, Lanconelli and Priola[1] adopted Coifman-Weiss' theorem to show the estimate (1.11) for $j = n$.
- When $n = 2$, in [2] we established a version of Fefferman-Stein's theorem and then used it to show the estimate (1.11) for $j = 1, 2$ even for nonlocal operators.

- Bramanti, Cupini, Lanconelli and Priola[1] adopted Coifman-Weiss' theorem to show the estimate (1.11) for $j = n$.
- When $n = 2$, in [2] we established a version of Fefferman-Stein's theorem and then used it to show the estimate (1.11) for $j = 1, 2$ even for nonlocal operators.
- In [2], we have used the following Bouchet's result[3]: Let u satisfy

$$\partial_t u + x_2 \cdot \nabla_{x_1} u = f,$$

then for any $\alpha \geqslant 0$,

$$\|\Delta_{x_1}^{\frac{\alpha}{2(1+\alpha)}} u\|_2 \leqslant C(\alpha, d) \|\Delta_{x_2}^{\frac{\alpha}{2}} u\|_2^{\frac{1}{1+\alpha}} \|f\|_2^{\frac{\alpha}{1+\alpha}}.$$

[1] M. Bramanti, G. Cupini, E. Lanconelli and E. Priola: Global L^p-estimate for degenerate Ornstein-Uhlenbeck operators. *Math Z.* **266** (2010), 789-816.

[2] Z.-Q. Chen and X. Zhang: L^p-maximal hypoelliptic regularity of nonlocal kinetic Fokker-Planck operator. *J. Math. Pures et Appliquées*, (2018+).

[3] F. Bouchut: Hypoelliptic regularity in kinetic equations. *J. Math. Pures Appl.* **81** (2002), 1135-1159.

- Consider the following nonlocal operator:

$$\widetilde{\mathscr{L}}_{\sigma}^{\nu} f(x) := \int_{\mathbb{R}^d} [f(x + \sigma y) + f(x - y) - 2f(x)]\nu(\mathrm{d}y),$$

where σ is a $d \times d$ matrix and ν is a symmetric Lévy measure.

- Consider the following nonlocal operator:

$$\widetilde{\mathscr{L}_{\sigma}^{\nu}} f(x) := \int_{\mathbb{R}^d} [f(x + \sigma y) + f(x - y) - 2f(x)] \nu(\mathrm{d}y),$$

  where σ is a $d \times d$ matrix and ν is a symmetric Lévy measure.
- Let $n \geqslant 2$ and

$$\mathscr{L}_t f(x) := \widetilde{\mathscr{L}_{\sigma_t, x_n}^{\nu_t}} f(x) + \sum_{j=2}^{n} x_j \cdot \nabla_{x_{j-1}} f(x),$$

  where $\widetilde{\mathscr{L}_{\sigma_t, x_n}^{\nu_t}}$ means that the operator acts on the variable x_n.

- Consider the following nonlocal operator:

$$\widetilde{\mathscr{L}}^{\nu}_{\sigma} f(x) := \int_{\mathbb{R}^d} [f(x+\sigma y) + f(x-y) - 2f(x)]\nu(\mathrm{d}y),$$

where σ is a $d \times d$ matrix and ν is a symmetric Lévy measure.

- Let $n \geqslant 2$ and

$$\mathscr{L}_t f(x) := \widetilde{\mathscr{L}}^{\nu_t}_{\sigma_t, x_n} f(x) + \sum_{j=2}^{n} x_j \cdot \nabla_{x_{j-1}} f(x),$$

where $\widetilde{\mathscr{L}}^{\nu_t}_{\sigma_t, x_n}$ means that the operator acts on the variable x_n.

- Suppose that

$$\|\sigma\|_\infty + \|\sigma^{-1}\|_\infty < \infty$$

and for some $\alpha \in (0,2)$,

$$\nu_1^{(\alpha)} \leqslant \nu_s \leqslant \nu_2^{(\alpha)},$$

where $\nu_1^{(\alpha)}$ and $\nu_2^{(\alpha)}$ are two symmetric and nondegenerate α-stable Lévy measures.

Theorem 2 (Nonlocal version)

Under the above assumptions, we have for any $j = 1, \cdots, n$,

$$\left\| \Delta_{x_j}^{\frac{\alpha}{2(1+\alpha(n-j))}} \int_0^\infty \mathrm{e}^{-\lambda t} \mathcal{T}_{s,t+s}^{\nu,\sigma} f(t+s,x) \mathrm{d}t \right\|_p \leqslant C\|f\|_p, \quad (1.12)$$

where $\mathcal{T}_{s,t}^{\nu,\sigma}$ *is defined as in* (1.8) *by using the time-inhomogeneous Markov process* $\{\{L_t^{s,x}; t \geqslant 0\}; (s,x) \in \mathbb{R} \times \mathbb{R}^{nd}\}$ *determined by the family of Lévy measures* $\{\nu_s, s \in \mathbb{R}\}$ *in place of Brownian motion.*

Remark: At the almost same time, Huang, Menozzi and Priola[1] obtained (1.12) for time-independent σ and ν by Coifman-Weiss' theorem.

[1] Huang L., Menozzi S. and Priola E.: L^p-estimates for degenerate non-local Kolmogorov operators. *J. Math. Pures et Appliquées*, (2018+).

Motivation

The first motivation comes from the study of Boltzmann equation.

- For $\mathbf{v}, \mathbf{v}_* \in \mathbb{R}^d$ and $\omega \in \mathbb{S}^{d-1}$, define

$$\mathbf{v}' = \mathbf{v} - \langle \mathbf{v} - \mathbf{v}_*, \omega \rangle \omega, \quad \mathbf{v}'_* = \mathbf{v}_* + \langle \mathbf{v} - \mathbf{v}_*, \omega \rangle \omega,$$

  where $\mathbf{v}, \mathbf{v}_*$ stand for the velocities of two particles before collision, and $\mathbf{v}', \mathbf{v}'_*$ stand for the velocities of two particles after collision, and ω stands for the angle of collision.

Motivation

The first motivation comes from the study of Boltzmann equation.

- For $\mathbf{v}, \mathbf{v}_* \in \mathbb{R}^d$ and $\omega \in \mathbb{S}^{d-1}$, define

$$\mathbf{v}' = \mathbf{v} - \langle \mathbf{v} - \mathbf{v}_*, \omega \rangle \omega, \ \ \mathbf{v}'_* = \mathbf{v}_* + \langle \mathbf{v} - \mathbf{v}_*, \omega \rangle \omega,$$

  where $\mathbf{v}, \mathbf{v}_*$ stand for the velocities of two particles before collision, and $\mathbf{v}', \mathbf{v}'_*$ stand for the velocities of two particles after collision, and ω stands for the angle of collision.
- $\mathbf{v}, \mathbf{v}_*$ and $\mathbf{v}', \mathbf{v}'_*$ satisfy the following momentum and energy conservations:

$$\mathbf{v} + \mathbf{v}_* = \mathbf{v}' + \mathbf{v}'_*, \ \ |\mathbf{v}|^2 + |\mathbf{v}_*|^2 = |\mathbf{v}'|^2 + |\mathbf{v}'_*|^2,$$

  and

$$\langle \mathbf{v}', \omega \rangle = \langle \mathbf{v}_*, \omega \rangle, \ \ \langle \mathbf{v}'_*, \omega \rangle = \langle \mathbf{v}, \omega \rangle.$$

- The classical inhomogeneous Boltzmann equation takes the following form

$$\partial_t f(t,x,\mathbf{v}) + \mathbf{v}\cdot\nabla_x f(t,x,\mathbf{v}) = Q(f,f)(t,x,\mathbf{v}),$$

where f stands for the density of the gas, and $Q(f,g)$ is the collision operator defined by

$$Q(f,g) := \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} (f(\mathbf{v}'_*)g(\mathbf{v}') - f(\mathbf{v}_*)g(\mathbf{v}))B(|\mathbf{v}-\mathbf{v}_*|,\omega)\mathrm{d}\omega\mathrm{d}\mathbf{v}_*,$$

where $B(|\mathbf{v}-\mathbf{v}_*|,\omega) = |\mathbf{v}-\mathbf{v}_*|^\gamma b(\langle \mathbf{v}-\mathbf{v}_*,\omega\rangle/|\mathbf{v}-\mathbf{v}_*|)$ and $b(s)\asymp s^{-1-\alpha}$, $\alpha\in(0,2)$ and $\gamma+\alpha\in(-1,1)$.

- Using the following elementary formula

$$\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} F(x,\omega)\mathrm{d}\omega\mathrm{d}x = \int_{\mathbb{R}^d}\int_{\{h\cdot w=0\}} F(h+w,\bar{w})|w|^{1-d}\mathrm{d}h\mathrm{d}w,$$

one can write the collision operator as the following form (Carleman's representation):

$$Q(f,g) = \int_{\mathbb{R}^d}\int_{\{h\cdot w=0\}}\Big[f(\mathbf{v}-h)g(\mathbf{v}+w) - f(\mathbf{v}-h-w)g(\mathbf{v})\Big] \times B(|h+w|, w/|w|)|w|^{1-d}\mathrm{d}h\mathrm{d}w.$$

- Using the following elementary formula

$$\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} F(x,\omega)\mathrm{d}\omega\mathrm{d}x = \int_{\mathbb{R}^d}\int_{\{h\cdot w=0\}} F(h+w,\bar{w})|w|^{1-d}\mathrm{d}h\mathrm{d}w,$$

one can write the collision operator as the following form (Carleman's representation):

$$Q(f,g) = \int_{\mathbb{R}^d}\int_{\{h\cdot w=0\}}\Big[f(\mathbf{v}-h)g(\mathbf{v}+w) - f(\mathbf{v}-h-w)g(\mathbf{v})\Big] \times B(|h+w|, w/|w|)|w|^{1-d}\mathrm{d}h\mathrm{d}w.$$

- In particular, if we let $b(s) = s^{-1-\alpha}$, then we can split Q into two parts

$$Q(f,g) = Q_1(f,g) + Q_2(f,g),$$

where $Q_1(f,g) := g(\mathbf{v})H_f(\mathbf{v})$ and

$$Q_2(f,g) := \int_{\mathbb{R}^d}(g(\mathbf{v}+w) - g(\mathbf{v}))\frac{K_f(\mathbf{v},w)}{|w|^{\alpha+d}}\mathrm{d}w,$$

- with

$$H_f(\mathbf{v}) := \int_{\mathbb{R}^d} \int_{\{h\cdot w=0\}} (f(\mathbf{v}-h) - f(\mathbf{v}-h-w)) \times |h+w|^{\gamma+1+\alpha}|w|^{\alpha-d}\mathrm{d}h\mathrm{d}w,$$

$$K_f(\mathbf{v}, w) := \int_{\{h\cdot w=0\}} f(\mathbf{v}-h)|h+w|^{\gamma+1+\alpha}\mathrm{d}h.$$

Notice that $K_f(\mathbf{v}, w) = K_f(\mathbf{v}, -w)$.

- with

$$H_f(\mathbf{v}) := \int_{\mathbb{R}^d} \int_{\{h\cdot w=0\}} (f(\mathbf{v}-h) - f(\mathbf{v}-h-w)) \times |h+w|^{\gamma+1+\alpha} |w|^{\alpha-d} \mathrm{d}h \mathrm{d}w,$$

$$K_f(\mathbf{v}, w) := \int_{\{h\cdot w=0\}} f(\mathbf{v}-h)|h+w|^{\gamma+1+\alpha} \mathrm{d}h.$$

Notice that $K_f(\mathbf{v}, w) = K_f(\mathbf{v}, -w)$.

- The linearized Boltzmann equation takes the following form:

$$\partial_t g + \mathbf{v}\cdot\nabla_x g = \int_{\mathbb{R}^d} (g(\mathbf{v}+w) - g(\mathbf{v})) \frac{K_f(\mathbf{v}, w)}{|w|^{\alpha+d}} \mathrm{d}w + g(\mathbf{v}) H_f(\mathbf{v}).$$

- with

$$H_f(\mathbf{v}) := \int_{\mathbb{R}^d} \int_{\{h\cdot w=0\}} (f(\mathbf{v}-h) - f(\mathbf{v}-h-w)) \times |h+w|^{\gamma+1+\alpha} |w|^{\alpha-d} \mathrm{d}h\mathrm{d}w,$$

$$K_f(\mathbf{v}, w) := \int_{\{h\cdot w=0\}} f(\mathbf{v}-h)|h+w|^{\gamma+1+\alpha}\mathrm{d}h.$$

Notice that $K_f(\mathbf{v}, w) = K_f(\mathbf{v}, -w)$.

- The linearized Boltzmann equation takes the following form:

$$\partial_t g + \mathbf{v}\cdot\nabla_x g = \int_{\mathbb{R}^d} (g(\mathbf{v}+w) - g(\mathbf{v})) \frac{K_f(\mathbf{v}, w)}{|w|^{\alpha+d}} \mathrm{d}w + g(\mathbf{v})H_f(\mathbf{v}).$$

[1] Villani C.: A review of mathematical topics in collisional kinetic theory. Handbook of Fluid Mechanics, 2002.

Propagation of L^2-regularity for transport equations

- Fix $n \geqslant 2$ and $\lambda > 0$. Let A be as in (1.3).

Propagation of L^2-regularity for transport equations

- Fix $n \geqslant 2$ and $\lambda > 0$. Let A be as in (1.3).
- Consider the following linear transport equation in $\mathbb{R}^{nd}$:

$$\partial_s u + Ax \cdot \nabla u - \lambda u + f = 0. \tag{1.13}$$

- Fix $n \geqslant 2$ and $\lambda > 0$. Let A be as in (1.3).
- Consider the following linear transport equation in $\mathbb{R}^{nd}$:

$$\partial_s u + Ax \cdot \nabla u - \lambda u + f = 0. \tag{1.13}$$

- Its solution is explicitly given by

$$\begin{aligned} u(s,x) &= -\int_0^\infty \partial_t\Big(\mathrm{e}^{-\lambda t} u\big(t+s, \mathrm{e}^{-tA}x\big)\Big)\mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{-\lambda t} f\big(t+s, \mathrm{e}^{-tA}x\big)\mathrm{d}t. \end{aligned} \tag{1.14}$$

Theorem 3

Let $f, u \in L^2(\mathbb{R}^{1+nd})$ so that(1.13) holds in the weak sense. For any $\alpha \geqslant 0$ and $j = 1, 2, \cdots, n-1$, there is a constant $C = C(\alpha, j, d) > 0$ such that

$$\|\Delta_{x_j}^{\frac{\alpha}{2(1+\alpha)}} u\|_2 \leqslant C\|\Delta_{x_{j+1}}^{\frac{\alpha}{2}} u\|_2^{\frac{1}{1+\alpha}} \|f\|_2^{\frac{\alpha}{1+\alpha}}. \tag{1.15}$$

In particular,

$$\|\Delta_{x_j}^{\frac{\alpha}{2(1+(n-j)\alpha)}} u\|_2 \leqslant C\|\Delta_{x_n}^{\frac{\alpha}{2}} u\|_2^{\frac{1+(n-j-1)\alpha}{1+(n-j)\alpha}} \|f\|_2^{\frac{\alpha}{1+(n-j)\alpha}}. \tag{1.16}$$

[1] F. Bouchut: Hypoelliptic regularity in kinetic equations. *J. Math. Pures Appl.* **81** (2002), 1135-1159.

[2] R. Alexander: Fractional order kinetic equations and hypoellipcity. *Anal. Appl.* (Singap.) **10**, no.3 (2012), 237-247.

- For any $r > 0$ and point $(t_0, x_0) \in \mathbb{R}^{1+nd}$, we introduce a family of “balls” in $\mathbb{R}^{1+nd}$:

$$Q_r(t_0, x_0) := \Big\{(t, x) : \ell(t - t_0, x - \mathrm{e}^{(t-t_0)A} x_0) \leqslant r\Big\},$$

where

$$\ell(t, x) := \max\Big\{|t|^{1/2}, |x_1|^{1/(2n-1)}, |x_2|^{1/(2n-3)}, \cdots, |x_{n-1}|^{1/3}, |x_n|\Big\}.$$

- For any $r > 0$ and point $(t_0, x_0) \in \mathbb{R}^{1+nd}$, we introduce a family of "balls" in $\mathbb{R}^{1+nd}$:

$$Q_r(t_0, x_0) := \left\{(t, x) : \ell(t - t_0, x - \mathrm{e}^{(t-t_0)A} x_0) \leqslant r\right\},$$

  where

$$\ell(t, x) := \max\left\{|t|^{1/2}, |x_1|^{1/(2n-1)}, |x_2|^{1/(2n-3)}, \cdots, |x_{n-1}|^{1/3}, |x_n|\right\}.$$

- We use $\mathbb{Q}$ to denote the set of all such balls.

- For any $r > 0$ and point $(t_0, x_0) \in \mathbb{R}^{1+nd}$, we introduce a family of "balls" in $\mathbb{R}^{1+nd}$:

$$Q_r(t_0, x_0) := \left\{(t, x) : \ell(t - t_0, x - \mathrm{e}^{(t-t_0)A} x_0) \leqslant r\right\},$$

where

$$\ell(t, x) := \max\left\{|t|^{1/2}, |x_1|^{1/(2n-1)}, |x_2|^{1/(2n-3)}, \cdots, |x_{n-1}|^{1/3}, |x_n|\right\}.$$

- We use $\mathbb{Q}$ to denote the set of all such balls.
- $\ell(r^2 t, \Theta_r x) = r\ell(t, x)$ for any $r > 0$. $|Q_r(t_0, x_0)| = \omega_d^n r^{n^2 d+2}$.

- For any $r > 0$ and point $(t_0, x_0) \in \mathbb{R}^{1+nd}$, we introduce a family of "balls" in $\mathbb{R}^{1+nd}$:

$$Q_r(t_0, x_0) := \left\{(t, x) : \ell(t - t_0, x - \mathrm{e}^{(t-t_0)A} x_0) \leqslant r\right\},$$

where

$$\ell(t, x) := \max\left\{|t|^{1/2}, |x_1|^{1/(2n-1)}, |x_2|^{1/(2n-3)}, \cdots, |x_{n-1}|^{1/3}, |x_n|\right\}.$$

- We use $\mathbb{Q}$ to denote the set of all such balls.
- $\ell(r^2 t, \Theta_r x) = r\ell(t, x)$ for any $r > 0$. $|Q_r(t_0, x_0)| = \omega_d^n r^{n^2 d+2}$.
- If $Q_r(t_0, x_0) \cap Q_r(t_0', x_0') \neq \emptyset$, then

$$Q_r(t_0, x_0) \subset Q_{20\cdot r}(t_0', x_0').$$

- For $f \in L^1_{loc}(\mathbb{R}^{1+nd})$, we define the Hardy-Littlewood maximal function by

$$\mathcal{M}f(t,x) := \sup_{r>0} \fint_{Q_r(t,x)} |f(t',x')| \mathrm{d}x' \mathrm{d}t',$$

- For $f \in L^1_{loc}(\mathbb{R}^{1+nd})$, we define the Hardy-Littlewood maximal function by

$$\mathcal{M}f(t,x) := \sup_{r>0} \fint_{Q_r(t,x)} |f(t',x')|\mathrm{d}x'\mathrm{d}t',$$

- The sharp function is defined by

$$\mathcal{M}^\sharp f(t,x) := \sup_{r>0} \fint_{Q_r(t,x)} |f(t',x') - f_{Q_r(t,x)}|\mathrm{d}x'\mathrm{d}t',$$

where for a $Q \in \mathbb{Q}$,

$$f_Q := \fint_Q f(t',x')\mathrm{d}x'\mathrm{d}t' := \frac{1}{|Q|}\int_Q f(t',x')\mathrm{d}x'\mathrm{d}t'.$$

- One says that a function $f \in BMO(\mathbb{R}^{1+nd})$ if $\mathcal{M}^\sharp f \in L^\infty(\mathbb{R}^{1+nd})$.

- One says that a function $f \in BMO(\mathbb{R}^{1+nd})$ if $\mathcal{M}^\sharp f \in L^\infty(\mathbb{R}^{1+nd})$.
- $f \in BMO(\mathbb{R}^{1+nd})$ if and only if there exists a constant $C > 0$ such that for any $Q \in \mathbb{Q}$, and for some $c_Q \in \mathbb{R}$,

$$\fint_Q |f(t', x') - c_Q| \mathrm{d}x' \mathrm{d}t' \leqslant C.$$

- One says that a function $f \in BMO(\mathbb{R}^{1+nd})$ if $\mathcal{M}^\sharp f \in L^\infty(\mathbb{R}^{1+nd})$.
- $f \in BMO(\mathbb{R}^{1+nd})$ if and only if there exists a constant $C > 0$ such that for any $Q \in \mathbb{Q}$, and for some $c_Q \in \mathbb{R}$,

$$\fint_Q |f(t', x') - c_Q| \mathrm{d}x' \mathrm{d}t' \leqslant C.$$

Theorem 4 (Fefferman-Stein-type theorem)

Suppose $q \in (1, \infty)$, and $\mathscr{P}$ is a bounded linear operator from $L^q(\mathbb{R}^{1+nd})$ to $L^q(\mathbb{R}^{1+nd})$ and also from $L^\infty(\mathbb{R}^{1+nd})$ to $BMO(\mathbb{R}^{1+nd})$. Then for any $p \in [q, \infty)$, there is a constant $C > 0$ depending only on p, q and the norms of $\|\mathscr{P}\|_{L^q \to L^q}$ and $\|\mathscr{P}\|_{L^\infty \to BMO}$ so that

$$\|\mathscr{P} f\|_p \leqslant C \|f\|_p \qquad \text{for every } f \in L^p(\mathbb{R}^{1+nd}).$$

Case: $p = 2$

For $f \in C_c^\infty(\mathbb{R}^{1+nd})$, by Fourier's transform and Hölder's inequality, we have

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\left\|\Delta_{x_n}\int_0^\infty \mathrm{e}^{-\lambda t}\mathcal{T}_{s,t+s}f(t+s,\cdot)\mathrm{d}t\right\|_2^2\mathrm{d}s\\
&=\int_{-\infty}^{\infty}\int_{\mathbb{R}^{nd}}|\xi_n|^2\left|\int_0^\infty \mathrm{e}^{-\lambda t}\widehat{\mathcal{T}_{s,t+s}f}(t+s,\xi)\mathrm{d}t\right|^2\mathrm{d}\xi\mathrm{d}s\\
&=\int_{-\infty}^{\infty}\int_{\mathbb{R}^{nd}}|\xi_n|^2\left|\int_0^\infty \mathrm{e}^{-\lambda t}\mathrm{e}^{-\frac{1}{2}\int_0^t|(\sigma^a_{s+r})^*\mathrm{e}^{-rA^*}\xi|^2\mathrm{d}r}\hat{f}(t+s,\mathrm{e}^{-tA^*}\xi)\mathrm{d}t\right|^2\mathrm{d}\xi\mathrm{d}s\\
&\leqslant\int_{-\infty}^{\infty}\int_{\mathbb{R}^{nd}}\left(\int_0^\infty|\xi_n|^2\mathrm{e}^{-\frac{1}{2}\int_0^t|(\sigma^a_{s+r})^*\mathrm{e}^{-rA^*}\xi|^2\mathrm{d}r}|\hat{f}(t+s,\mathrm{e}^{-tA^*}\xi)|^2\mathrm{d}t\right)\\
&\qquad\times\left(\int_0^\infty|\xi_n|^2\mathrm{e}^{-\frac{1}{2}\int_0^t|(\sigma^a_{s+r})^*\mathrm{e}^{-rA^*}\xi|^2\mathrm{d}r}\mathrm{d}t\right)\mathrm{d}\xi\mathrm{d}s.
\end{aligned}
$$

- One can show easily

$$\left(\int_0^t |(\sigma^a_{s+r})^* \mathrm{e}^{-rA^*}\xi|^2 \mathrm{d}r\right) \wedge \left(\int_0^t |(\sigma^a_{s+r})^* \mathrm{e}^{(t-r)A^*}\xi|^2 \mathrm{d}r\right) \geqslant c|\Theta_{t^{1/2}}\xi|^2,$$

- One can show easily

$$\left(\int_0^t |(\sigma^a_{s+r})^* \mathrm{e}^{-rA^*}\xi|^2 \mathrm{d}r\right) \wedge \left(\int_0^t |(\sigma^a_{s+r})^* \mathrm{e}^{(t-r)A^*}\xi|^2 \mathrm{d}r\right) \geqslant c|\Theta_{t^{1/2}}\xi|^2,$$

- From these two estimates, we have

$$\int_0^\infty |\xi_n|^2 \mathrm{e}^{-\frac{1}{2}\int_0^t |(\sigma^a_{s+r})^* \mathrm{e}^{-rA^*}\xi|^2 \mathrm{d}r} \mathrm{d}t \leqslant c$$

and

$$\int_0^\infty |(\mathrm{e}^{tA^*}\xi)_n|^2 \mathrm{e}^{-\frac{1}{2}\int_0^t |(\sigma^a_{s-t+r})^* \mathrm{e}^{(t-r)A^*}\xi|^2 \mathrm{d}r} \mathrm{d}t \leqslant c$$

By the change of variables and Fubini's theorem, we obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\left\|\Delta_{x_n}\int_0^\infty \mathrm{e}^{-\lambda t}\mathcal{T}_{s,t+s}f(t+s,\cdot)\mathrm{d}t\right\|_2^2\mathrm{d}s\\
&\leqslant c^{-1}\int_{-\infty}^{\infty}\int_{\mathbb{R}^{nd}}\left(\int_0^\infty |\xi_n|^2\mathrm{e}^{-\frac{1}{2}\int_0^t|(\sigma^a_{s+r})^*\mathrm{e}^{-rA^*}\xi|^2\mathrm{d}r}|\hat{f}(t+s,\mathrm{e}^{-tA^*}\xi)|^2\mathrm{d}t\right)\mathrm{d}\xi\mathrm{d}s\\
&=c^{-1}\int_{-\infty}^{\infty}\int_{\mathbb{R}^{nd}}\left(\int_0^\infty |(\mathrm{e}^{tA^*}\xi)_n|^2\mathrm{e}^{-\frac{1}{2}\int_0^t|(\sigma^a_{s+r})^*\mathrm{e}^{(t-r)A^*}\xi|^2\mathrm{d}r}|\hat{f}(t+s,\xi)|^2\mathrm{d}t\right)\mathrm{d}\xi\mathrm{d}s\\
&=c^{-1}\int_{-\infty}^{\infty}\int_{\mathbb{R}^{nd}}\left(\int_0^\infty |(\mathrm{e}^{tA^*}\xi)_n|^2\mathrm{e}^{-\frac{1}{2}\int_0^t|(\sigma^a_{s-t+r})^*\mathrm{e}^{(t-r)A^*}\xi|^2\mathrm{d}r}\mathrm{d}t\right)|\hat{f}(s,\xi)|^2\mathrm{d}\xi\mathrm{d}s\\
&\leqslant 2c^{-2}\int_{-\infty}^{\infty}\int_{\mathbb{R}^{nd}}|\hat{f}(s,\xi)|^2\mathrm{d}\xi\mathrm{d}s=2c^{-2}\|f\|_2^2.
\end{aligned}
$$

This together with (1.16) completes the proof of (1.11) for $p=2$.

Case: $p \in (2, \infty)$

- Let $\varrho \in C_c^\infty(\mathbb{R}^{nd})$ be nonnegative with $\int \varrho = 1$. Define

$$\varrho_\varepsilon(x) = \varepsilon^{-nd} \varrho(x/\varepsilon); \quad \varepsilon > 0,$$

and for a function $f(t, x)$ defined on $\mathbb{R} \times \mathbb{R}^{nd}$ and $\varepsilon > 0$, let

$$f_\varepsilon(t, x) := f(t, \cdot) * \varrho_\varepsilon(x) := \int_{\mathbb{R}^d} f(t, y) \varrho_\varepsilon(x - y) \mathrm{d}y.$$

Case: $p \in (2, \infty)$

- Let $\varrho \in C_c^\infty(\mathbb{R}^{nd})$ be nonnegative with $\int \varrho = 1$. Define

$$\varrho_\varepsilon(x) = \varepsilon^{-nd}\varrho(x/\varepsilon); \quad \varepsilon > 0,$$

and for a function $f(t, x)$ defined on $\mathbb{R} \times \mathbb{R}^{nd}$ and $\varepsilon > 0$, let

$$f_\varepsilon(t, x) := f(t, \cdot) * \varrho_\varepsilon(x) := \int_{\mathbb{R}^d} f(t, y)\varrho_\varepsilon(x - y)\mathrm{d}y.$$

- For $j = 1, \cdots, n$ and $\varepsilon \in (0, 1)$, define

$$\mathscr{P}_j^\varepsilon f := \mathscr{P}_j^a f_\varepsilon(s, x) := \Delta_{x_j}^{1/(1+2(n-j))} \int_0^\infty \mathrm{e}^{-\lambda t}\mathcal{T}^a_{s,t+s} f_\varepsilon(t + s, x)\mathrm{d}t,$$

where the superscript a denotes the dependence on the diffusion coefficient a.

- Our main task is to show that $\mathscr{P}_j^\varepsilon$ is a bounded linear operator from $L^\infty(\mathbb{R}^{1+nd})$ to BMO. More precisely, we want to prove that for any $f \in L^\infty(\mathbb{R}^{1+nd})$ with $\|f\|_\infty \leqslant 1$, and any $Q = Q_r(t_0, x_0) \in \mathbb{Q}$,

$$⨍_Q |\mathscr{P}_j^a f_\varepsilon(s,x) - c_j^Q|^2 \leqslant C, \tag{1.17}$$

where c_j^Q is a constant depending on Q and f_ε, and C only depends on n, κ, p, d, and not on ε.

- Our main task is to show that $\mathscr{P}_j^\varepsilon$ is a bounded linear operator from $L^\infty(\mathbb{R}^{1+nd})$ to *BMO*. More precisely, we want to prove that for any $f \in L^\infty(\mathbb{R}^{1+nd})$ with $\|f\|_\infty \leqslant 1$, and any $Q = Q_r(t_0, x_0) \in \mathbb{Q}$,

$$\fint_Q |\mathscr{P}_j^a f_\varepsilon(s,x) - c_j^Q|^2 \leqslant C, \tag{1.17}$$

where c_j^Q is a constant depending on Q and f_ε, and C only depends on n, κ, p, d, and not on ε.

Lemma 5 (Scaling Property)

For any $Q = Q_r(t_0, x_0) \in \mathbb{Q}$ *, we have*

$$\fint_{Q_r(t_0,x_0)} \left|\mathscr{P}_j^a f_\varepsilon(s,x) - c\right|^2 = \fint_{Q_1(0)} \left|\mathscr{P}_j^{\tilde{a}} \tilde{f}_\varepsilon(s,x) - c\right|^2, \tag{1.18}$$

where $c \in \mathbb{R}$, $\tilde{a}_s := a_{r^2 s + t_0}$ *and* $\tilde{f}_\varepsilon(t,x) := f_\varepsilon\Big(r^2 t + t_0, \Theta_r x + \mathrm{e}^{tA} x_0\Big)$*. Here* Θ_r *is the dilation operator defined in* (1.7)*.*

Lemma 6

Under (1.2), $X_t^{s,\mathbf{0}}$ of (1.6) has a smooth density function $p_{s,t}^{(\mathbf{0})}(y)$. For each $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, there are constants $C, c > 0$ only depending on n, β, d and κ such that for all $s < t$ and $x, y \in \mathbb{R}^{nd}$,

$$|\nabla_{y_1}^{\beta_1} \cdots \nabla_{y_n}^{\beta_n} p_{s,t}^{(\mathbf{0})}(y)| \leqslant C(t-s)^{-(n^2 d + \sum_{i=1}^n (2(n-i)+1)\beta_i)/2} \mathrm{e}^{-c|\Theta_{(t-s)^{-1/2}} y|^2},$$

where Θ_r is the dilation operator defined by (1.7).

Lemma 6

Under (1.2), $X_t^{s,\mathbf{0}}$ *of* (1.6) *has a smooth density function* $p_{s,t}^{(\mathbf{0})}(y)$. *For each* $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{N}_0^n$, *where* $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, *there are constants* $C, c > 0$ *only depending on* n, β, d *and* κ *such that for all* $s < t$ *and* $x, y \in \mathbb{R}^{nd}$,

$$|\nabla_{y_1}^{\beta_1} \cdots \nabla_{y_n}^{\beta_n} p_{s,t}^{(\mathbf{0})}(y)| \leqslant C(t-s)^{-(n^2 d + \sum_{i=1}^n (2(n-i)+1)\beta_i)/2} \mathrm{e}^{-c|\Theta_{(t-s)^{-1/2}} y|^2},$$

where Θ_r *is the dilation operator defined by* (1.7).

Corollary 7

For any $j = 1, \cdots, n$, $\alpha \in (0, 2]$ *and* $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{N}_0^n$, *there is a constant* $C > 0$ *such that for all* $f \in C_b^\infty(\mathbb{R}^{nd})$ *and* $s < t$,

$$\|\Delta_{x_j}^{\alpha/2} \nabla_{x_1}^{\beta_1} \cdots \nabla_{x_n}^{\beta_n} \mathcal{T}_{s,t} f\|_\infty \leqslant C|t-s|^{-(\sum_{i=1}^n (2(n-i)+1)\beta_i + (2(n-j)+1)\alpha)/2} \|f\|_\infty,$$

$$\|\nabla_{x_1}^{\beta_1} \cdots \nabla_{x_n}^{\beta_n} \mathcal{T}_{s,t} \Delta_{x_j}^{\alpha/2} f\|_\infty \leqslant C|t-s|^{-(\sum_{i=1}^n (2(n-i)+1)\beta_i + (2(n-j)+1)\alpha)/2} \|f\|_\infty,$$

where $\Delta_{x_j}^{\alpha/2}$ *means that the fractional Laplacian acts on the variable* x_j.

Proof of (1.11) for $p \in (2, \infty)$.

By the scaling property, the above Corollary and the well-proved estimate for $p = 2$, one can show $\mathscr{P}_j^\varepsilon : L^\infty(\mathbb{R}^{1+nd}) \to BMO$ is a bounded linear operator with bound independent of ε. Estimate (1.11) for $p \in (2, \infty)$ follows by Theorem 4 and the well-proved estimate for $p = 2$. □

Proof of (1.11) for $p \in (2, \infty)$.

By the scaling property, the above Corollary and the well-proved estimate for $p = 2$, one can show $\mathscr{P}_j^\varepsilon : L^\infty(\mathbb{R}^{1+nd}) \to BMO$ is a bounded linear operator with bound independent of ε. Estimate (1.11) for $p \in (2, \infty)$ follows by Theorem 4 and the well-proved estimate for $p = 2$. □

Proof of (1.11) for $p \in (1, 2)$.

It follows by a duality argument. □

Thank you very much for your kind attention!